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The Zeros of Rational Splines and Complex Splines

HAN-LIN CHEN

Institute of Mathematics, Academia Sinica, Beijing, Peoples Republic of China

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INTRODUCTION

Many mathematical and physical problems are concerned with the interpolation of finite sets of data by certain functions, or with looking for a suitable function to approximate a function about which only little information is given. In many cases one uses polynomial spline functions, but in many other situations, for instance if the interpolated function has some singularities and is regular at infinity, polynomial spline functions are ineffective. Then rational functions or rational spline functions may be more suitable. If there are a lot of data points and we want the denominator and the numerator of the rational function to be polynomials of low degree, then rational spline functions are more efficient than rational functions.

A basic problem related to the theory of interpolation is that of estimating the total number of zeros of a function. In Section 1 of this paper, we obtain some results about a certain class of real analytic functions and rational splines (Theorems 1 and 2).

As for real spline functions, many authors have been engaged in this work and many results have been obtained (see [8, 10–12, 15]).

Due to the many papers about complex splines (see [1–7, 9, 13, 14, 16]), we have obtained a deeper understanding of how the complex splines play a special role in the theory of approximation (of analytic or pseudo-analytic functions, etc.), but very little has been published about the zeros of complex spline functions. In fact, till now, the fundamental theorem of algebra for complex splines has not been established. We attempt to explore this problem in Section 2. In connection with Section 1, we obtain a sharp upper bound of the zeros of a certain class of complex splines (Theorem 3).

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1. The Zeros of a Certain Class of Real Analytic Functions and Rational Spline Functions

Let $g(x)$ be a real analytic function defined on $(a, b) \subset E_1 = (-\infty, +\infty)$. We assume that a, b are not the limit points of the zeros of $g^{(i)}(x)$ $(i \leqslant m)$. The family of all such functions is denoted by $A(a, b)$.

Definition 1. The class $G_m(a, b)$ is the subclass of those $g \in A(a, b)$ which satisfy the following conditions: If for some $\bar{x} \in (a, b)$, $g^{(m)}(\bar{x}) = 0$, $g^{(m-1)}(\bar{x}) \neq 0$, then $g^{(m)}(x)$ has a zero at $\bar{x}$ of multiplicity α, α is an even number and $g^{(m-1)}(\bar{x})\, g^{(m+\alpha)}(\bar{x}) < 0$.

We list some examples.

Example 1. $g(x) \in A(a, b)$, $g^{(m)}(x) > 0$ (or $g^{(m)}(x) < 0$) for all $x \in (a, b)$. All the polynomials of exact degree m are contained in this class.

Example 2. $g(x)$ is the solution of the following differential equation

$$g^{(m)}(x) - (x - \bar{x})^{\alpha} f(x) = 0, \qquad \alpha \geqslant 0, \qquad \alpha \text{ even}.$$

Here $\bar{x} \in (a, b)$, and $g(x)$ satisfies the condition $g^{(m-1)}(\bar{x}) f^{(\alpha)}(\bar{x}) < 0$ at the point $\bar{x}$, where $f(x)$ (>0) is a real analytic function on (a, b).

Let $Z_m(g; (a, b))$ denote the total number of zeros of $g(x)$ in (a, b) counting multiplicities, all the multiplicities considered are less than or equal to m.

$S^+(c_i)_0^m$ denotes the maximum number of sign changes in the ordered sequence $c_0, \ldots, c_m$ when each zero entry is allowed to be $+1$ or -1.

We now extend the Budan Fourier theorem to the class $G_m(a, b)$.

Theorem 1. *If $g(x) \in G_m(a, b)$, then*

$$Z_m(g; (a, b)) = m - S^+((-1)^j g^{(j)}(a + \varepsilon))_0^m - S^+(g^{(j)}(b - \varepsilon))_0^m - H,$$

where $H \geqslant 0$ (H an integer), and ε is a small positive number.

Remark. If $g(x)$ is a polynomial of degree m, then we have the classical Budan Fourier theorem for polynomials; in this case $H \geqslant 0$ is an even number.

Proof. Since a, b are not the limit points of the zeros of $g^{(i)}(x)$ $(c_0 \leqslant i \leqslant m)$, $g^{(i)}(x)$ is a real analytic function, and it has only a finite number of zeros. Let ξ_j $(j = \overline{1, \beta})$ be all the distinct points in (a, b) in their natural order at which $g(x)$ or any of its ith derivatives $(i \leqslant m)$ vanish. Furthermore, set $\xi_0 = a$, $\xi_{\beta+1} \equiv b$.

Let $W(\xi_i)$ and $W_2(\xi_1)$ be the following numbers:

$$W(\xi_i) = S^+((-1)^j g^{(j)}(\xi_i))_0^m + S^+(g^{(j)}(\xi_i))_0^m - m,$$
$$W_\varepsilon(\xi_i) = S^+(g^{(j)}(\xi_{k_i} - \varepsilon))_0^m + S^+((-1)^j g^{(j)}(\xi_{k_i} + \varepsilon))_0^m - m.$$

Here $\{\xi_{k_i}\}_{i=1}^l$ are the distinct zeros of $g^{(m)}(x)$ and ε is a small positive number.

By using Taylor's expansion of $g(x)$ near the point ξ_{k_i}, $i = 1, \ldots, l$, we can prove

$$W_\varepsilon(\xi_{k_i}) = \alpha_{k_i} + h_{k_i}, \qquad h_{k_i} \geqslant 0, \qquad i = 1, \ldots, l, \tag{1}$$

and if $\xi_i \neq \xi_{k_j}$ $(j = 1, \ldots, l)$, then (see [8])

$$W(\xi_i) = \alpha_i + h_i, \qquad h_i \geqslant 0; \qquad h_i \text{ even}. \tag{2}$$

Let I denote the set of integers $\{k_1, \ldots, k_l\}$. It is easy to prove that

$$\sum_{i \notin I} W(\xi_i) + \sum_{i \in I} W_\varepsilon(\xi_i)$$
$$= m - S^+((-1)^j g^{(j)}(a + \varepsilon))_0^m - S^+(g^{(j)}(b - \varepsilon))_0^m. \tag{3}$$

Theorem 1 follows now from (1), (2), (3). Q.E.D.

DEFINITION 2. The families $CT(r, n)$, $RT(r, n)$, $G_{n,\lambda}(E_1)$ are defined as follows. Let $CT(r, n)$ denote the family of complex splines defined on the real x-axis with knots $\{x_j\}_{j=1}^r$; if $T(x) \in CT(r, n)$ then

$$T(x) = g_n(x) + \sum_{k=1}^r d_k(x - x_k)_+^n, \qquad \sum_{k=1}^r d_k(x - x_k)^n \equiv 0, \qquad x \in E_1, \tag{4}$$

where $g_n(x)$ is a complex polynomial of degree n, $\{d_k\}$ being complex constants. Let $RT(r, n)$ denote the family of all real splines in $CT(r, n)$. $G_{n,\lambda}(E)$ is the family of all rational functions

$$R(x) = \frac{T(x)}{(x - \lambda)^n}, \qquad T(x) \in RT(r, n), \qquad T(\lambda) \neq 0,$$

such that $R(x) \in G_n(\gamma_j)$ if $R(x) \not\equiv 0$, $x \in \gamma_j$, $j = \overline{(0, r+1)}$, where $\gamma_j = [\tau_j, \tau_{j+1})$, $\gamma_0 = (-\infty, \tau_1)$, $\gamma_{r+1} = [\tau_{r+1}, +\infty)$. $\{\tau_1, \ldots, \tau_{r+1}\}$ is the rearrangement of $\{\lambda, x_1, \ldots, x_r\}$ in the natural order.

We now follow Schumacher [15] and define the multiplicities of zeros of $T(x)$ on $(-\infty, +\infty)$ in the usual way (see [10]).

The distinct points give the following cases:

If $T(x) \equiv 0$, $x \in (-\infty, x_1) \cup [x_r, +\infty)$, define $Z(T, (-\infty, x_1) \cup [x_r, \infty)) = n+1$. If $T(x)$ is a polynomial of degree m ($m \leqslant n$) on $(-\infty, x_1) \cup [x_r, +\infty)$, then we say $T(x)$ has a zero at infinity with multiplicity $n-m$.

For later use, we list the following Lemma 1 given by Melkman (see [10]); it also easily follows from the proof of Theorem 1.

LEMMA 1. *Let $T(x)$ be a function of $RT(r, n)$, $(a, b) \subset E_1$, let there be k ($k \geqslant n+2$) knots in (a, b) denoted by $\{\xi_j\}_1^k$, then*

$$Z(T; (a, b)) = n_a + k - S^+((-1)^l\, T^{(l)}(a+))_0^{n_a}$$
$$- S^+(T^{(l)}(b-))_0^{n_b} - Zh - \sum_{i=1}^{k} W_i,$$

where $W_j = S^+((-1)^l\, T^{(l)}(\xi_j+))_0^{n_j} + S^+(T^{(l)}(\xi_j-))^{n_{j-1}} - n_j - \alpha_j + 1$, α_j is the multiplicity of the zero ξ_j. Here n_j, n_a, n_b are the degrees of $T(x)$ on (ξ_j, ξ_{j+1}), (a, ξ_1) and (ξ_k, b), respectively, and $h \geqslant 0$, $W_i \geqslant 0$.

From Lemma 1 we have

LEMMA 2. *If $T(x) \in RT(r, n)$, then*

(i) *$Z(T, E_1) \leqslant n_0 + r$, where r is the number of knots in E_1, n_0 is the degree of $T(x)$ on $\gamma_0 \cup \gamma_r$.*

(ii) *If $T(x) \equiv 0$, $x \bar{\in} (x_1, x_k)$, then*

$$Z(T, (x_1, x_k)) \leqslant k - n - 2, \qquad Z(T, E_1) \leqslant k - 1.$$

(iii) *If $n = 1$, then*

$$Z(T, E_1) \leqslant r, \qquad r \text{ odd},$$
$$\leqslant r - 1, \qquad r \text{ even}.$$

THEOREM 2. *Let $R(x) = T(x)/(x-\lambda)^n \in G_{n,\lambda}(E_1)$, and suppose that $T(x)$ has the form*

$$T(x) = \sum_{i=0}^{n} A_i x^i \qquad (x \in \gamma_0 \cup \gamma_r), \qquad A_{n-1} + \lambda n A_n \neq 0.$$

(i) *Case λ is not a knot: If $A_n \neq 0$, then*

$$Z(R, E_1) \leqslant r \qquad \text{if} \quad r, n \text{ odd or } r, n \text{ even},$$
$$\leqslant r + 1 \qquad \text{if} \quad r \text{ odd (even) and } n \text{ even (odd)}.$$

If $A_n = 0$, $A_{n-1} \neq 0$, *then*

$$Z(R, E_1) = r \qquad \text{if} \quad r \text{ odd (even) and } n \text{ even (odd)},$$
$$\leqslant r - 1 \qquad \text{if} \quad r, n \text{ odd or } r, n \text{ even}.$$

(ii) *Case* λ *is a knot: If* $A_n \neq 0$, *then*

$$Z(R, E_1) \leqslant r \qquad \text{if} \quad r, n \text{ odd or } r, n \text{ even},$$
$$\leqslant r - 1 \qquad \text{if} \quad r \text{ odd (even) and } n \text{ even (odd)}.$$

If $A_n = 0$, *then*

$$A(R, E_1) \leqslant r - 1 \qquad \text{if} \quad r, n \text{ odd or } r, n \text{ even},$$
$$\leqslant r - 2 \qquad \text{if} \quad r \text{ odd (even) and } n \text{ even (odd)}.$$

Proof. The lth derivative of $R(x)$ may be written as

$$R^{(l)}(x) = \frac{(-1)^l}{(n-1)!} \sum_{k=0}^{l} (-1)^k (n + l - k - 1)! \binom{l}{k} \frac{T^{(k)}(x)}{(x-\lambda)^{n+l-k}}, \tag{5}$$

where $x \neq \lambda$.

From (5) and $T(\lambda) \neq 0$ we see that if $T(x)$ has a zero of multiplicity α at $\bar{x}$, then $R(x)$ has a zero at the same point with the same multiplicity, and vice versa. If $T(x) \equiv 0$ on some interval γ_j, we may apply a linear transformation and use Lemma 2, noting that $T(\lambda) \neq 0$, to confirm that the above conclusions are valid.

Assume $T(x) \not\equiv 0$, $x \in \gamma_j$ $j = \overline{0, r+1}$.

Let $W_\varepsilon(x_j) = S^+(R^{(i)}(x_j - \varepsilon))_0^n + S^+((-1)^i R^{(i)}(x_j + \varepsilon))_0^n - n$. From the Taylors expansion of $R(x)$, we infer that $W_\varepsilon(x_j) = \alpha_j + h_j$, $h_j \geqslant 0$, if $R^{(n)}(x_j-) = R^{(n)}(x_j+) = 0$ (or $R^{(n)}(x_j-) = 0$, $R^{(n)}(x_j+) \neq 0$), and $W_\varepsilon(x_j) = \alpha_j - 1 + h_j$, $h_j \geqslant 0$, if $R^{(n)}(x_j+) = 0$, $R^{(n)}(x_j-) \neq 0$ (or $R^{(n)}(x_j-) \neq 0$, $R^{(n)}(x_j+) \neq 0$).

Let $I_0 = (-\infty, \lambda)$, $I_1 = (\infty, \lambda)$; by using Theorem 1 we obtain

$$Z(R, I_0 \cup I_1) = 2n + r - S^+((-1)^l R^{(l)}(-N))_0^n - S^+(R^{(l)}(N))_0^n$$
$$-S^+(R^{(l)}(\lambda - \varepsilon))_0^n - S^+((-1)^l R^{(l)}(\lambda + \varepsilon)_0^n - \sum h_j - \sum H_j,$$

where $h_j \geqslant 0$, $H_j \geqslant 0$. It is easy to prove that

$$S^+((-1)^l R^{(l)}(\lambda + \varepsilon))_0^n + S^+(R^{(l)}(\lambda - \varepsilon))_0^n = 0,$$

for sufficiently small ε, $\varepsilon > 0$. For sufficiently large $|x|$, we have

$$\begin{aligned} \operatorname{sign} R^{(l)}(x) &= \operatorname{sign}[(-1)^l x^{l+1}(A_{n-1} + n\lambda A_n)], && l \geqslant 1, \\ &= \operatorname{sign} A_n, \quad A_n \neq 0, && l = 0, \\ &= \operatorname{sign}(xA_{n-1}), \quad A_n = 0, && l = 0. \end{aligned}$$

From the expressions above and $Z(T, E_1) = Z(R, E_1)$, Theorem 2 follows from the fact that the definition of zeros of $T(x)$ is chosen so that $T(x)$ changes sign if $Z(T, \bar{x})$ $(=\alpha)$ is odd, and does not change sign if $Z(T, \bar{x})$ is even.

2. The Zeros of a Certain Class of Complex Spline Functions

Let Γ be the unit circle, Δ_z is a set of points $\{Z_j\}_1^r$ arranged in counterclockwise order on Γ. Let Γ_j be the circular arc $Z_j Z_{j+1} = \{Z \mid Z = e^{i\theta}, \theta_j < \theta < \theta_{j+1}\}$, $Z_j = e^{i\theta_j}$, $Z_{r+1} = Z_1$, $\theta_{r+1} = \theta_1 + 2\pi$.

$\mathscr{S}(n)$ is the family of complex polynomial splines of degree n defined on Γ with exactly r knots; if $S(Z) \in \mathscr{S}(n)$, then $S(Z) \in C^{n-1}(\Gamma)$.

If $\xi \in \Gamma \backslash \Delta_Z$, then we may define the multiplicity α_ξ of the zero of $S(Z) \in \mathscr{S}(n)$ at the point ξ in the usual way.

But near any point $\xi \in \Delta_Z$, $S(Z)$ can be written as

$$\begin{aligned} S(Z) &= C_{j-1}(Z - Z_j)^n, \quad Z \in \Gamma_{j-1}, \quad C_{j-1} \neq 0, \\ &= C_j(Z - Z_j)^n, \quad Z \in \Gamma_j, \quad C_j \neq 0. \end{aligned}$$

Let a_φ, b_φ be two real functions of φ:

$$a_\varphi = \operatorname{Re}(\varphi C_{j-1}) \neq 0, \qquad b_\varphi = \operatorname{Re}(\varphi C_j) \neq 0, \tag{6}$$

where $\varphi \neq 0$ is a complex number; if $a_\varphi b_\varphi < 0$ is valid for all complex $\varphi (\varphi \neq 0)$ satisfying (6), we then define ξ $(=Z_j)$ to be a zero of $S(Z)$ of multiplicity $n+1$, otherwise n.

If $S(Z) \equiv 0$ for $Z \in \Gamma_j$ then we say $S(Z)$ has a zero interval and count it as a zero of $S(Z)$ of multiplicity $n+1$.

Let $l_\omega(x)$ be a linear transformation $Z = \omega(x - i)/(x + i)$, where x is a real variable, i is the imaginary unit, ω is a point on Γ_r. The set $\Delta_Z = \{Z_j\}_1^r$ is transformed onto the set $\Delta_x = \{x_j\}_1^r$ by l_ω^{-1} such that $l_\omega : E_1 \to \Gamma$,

$$-\infty < x_1 < \cdots < x_r < +\infty.$$

If $\omega = Z_1$, then the point Z_1 corresponds to infinity, and $\{Z_j\}_2^r$ are converted to $\{x_j\}_2^r$.

Now we define a function $T_\omega(x)$ as follows,

$$T_\omega(x) = \mathrm{Re}\{\overline{S^{(\alpha)}(\omega)}(-2\omega i)^{-\alpha}(x+i)^n S(Z(x))\}, \tag{7}$$

where α is the multiplicity of the zero ω of $S(Z)$, $\alpha \geqslant 0$; in a neighbourhood of the point ω, $S(Z)$ can be written as $S(Z) = (Z-\omega)^\alpha g(Z)$, $g(\omega) \neq 0$, $g(Z) = \sum_0^{n-\alpha} b_l Z^l$. Since $Z - \omega = -2i\omega/(x+i)$, $S^{(\alpha)}(\omega) = \alpha!\, g(\omega)$, we have $T_\omega = \alpha!\, \mathrm{Re}\{\overline{g(\omega)} \sum_{l=0}^{n-\alpha} b_l \omega^l (x-i)^l (x+i)^{n-\alpha-l}\}$, for $|x|$ sufficiently large. We see that the main term in $T_\omega(x)$ is $x^{n-\alpha} (x \in 0_\sigma)$, its coefficient is $\alpha!\, |g(\omega)|^2 > 0$, therefore $T_\omega(x) \not\equiv 0$, $x \in E_1$. Since $S(Z) \in \mathscr{S}(n)$, it is easy to verify that $T_\omega(x)$ belongs to $RT(r, n)$.

If $\bar{Z} \neq \omega$ is a zero of $S(Z)$ of multiplicity β and $\bar{x} = l_\omega^{-1}(\bar{Z})$ is the corresponding point in E_1, it is not hard to see that $T_\omega(x)$ has a zero at $\bar{x}$ of multiplicity at least β. Therefore

$$Z(S, \Gamma \mid \omega) \leqslant Z(T_\omega, E_1). \tag{8}$$

We define a subclass $G_n(\Gamma) \subseteq \mathscr{S}(n)$ as follows. $S(Z) \in G_n(\Gamma)$ if $R(x) = T_\omega(x)/(x-\lambda)^n \in G_{n,\lambda}(E_1)$ for any ω and λ. We then have

THEOREM 3. *Let $S(Z) \in G_n(\Gamma)$. Then one has*

$$Z(S, \Gamma) \leqslant r, \quad \text{if } n, r \text{ odd or } n, r \text{ even},$$
$$\leqslant r - 1, \quad \text{otherwise}.$$

If $S(Z)$ has a simple zero ξ, ξ is a knot, then $Z(S, \Gamma) \leqslant r - 1$. If $S(Z)$ has a simple zero ξ, ξ is not a knot, then

$$Z(S, \Gamma) \leqslant r, \quad \text{if } n, r \text{ odd}$$
$$\leqslant r - 1, \quad \text{otherwise}.$$

The proof of this theorem depends on some lemmas.

Suppose $S(Z) \equiv 0$, $Z \in \Gamma_j$, but $S(Z) \not\equiv 0$, $Z \in \Gamma$. With no loss of generality, we may assume $j = r$. By using a linear transformation $l_\omega(x)$, $\omega \in \Gamma_r$, we obtain a function $\hat{S}(x) = (x+i)^n S(Z(x))$ belonging to $CT(r, n)$ such that either the real part of $\hat{S}(x)$ or the imaginary part cannot vanish identically on E_1. Denote one of them by $T(x)$, $T(x) \not\equiv 0$, $x \in E_1$, but $T(x) \equiv 0$, $x \in \gamma_0 \cup \gamma_r$. From Lemma 2, (ii), and (8), we infer $Z(S, \Gamma \mid \Gamma_r) \leqslant Z(T; (x_1, x_r)) \leqslant r - n - 2$. In view of the definition of the zero interval of $S(Z)$, we obtain $Z(S, \Gamma) \leqslant r - 1$. We have the following

LEMMA 3. *Let $S(Z) \in \mathscr{S}(n)$, if $S(Z) \not\equiv 0$, $Z \in \Gamma$, but for some k, $S(Z) \equiv 0$, $Z \in \Gamma_k$; then $Z(S, \Gamma) \leqslant r - 1$.*

From now on, we assume that $S(Z)$, $T_\omega(x)$ and $R(x)$ have no zero intervals.

If $S(Z) \in \mathscr{S}(n)$ and r (the number of knots) $\leqslant n+1$, then $S(Z)$ will be a polynomial of degree n; thus, later on, we only consider the case $r \geqslant n+2$.

LEMMA 4. *Let $S(Z) \in \mathscr{S}(n)$, $S(Z)$ have the following form on Γ_r, $S(Z) = a_n Z^n + \cdots + a_0$, $Z \in \Gamma_r$, $a_n \neq 0$. Then we can find a point ω on Γ_r such that*

$$S(\omega) \neq 0, \qquad \operatorname{Im}\{\omega \overline{S(\omega)}\, S'(\omega))\} = 0. \tag{9}$$

From Lemma 4, $A_n = |S(\omega)|^2 > 0$, $A_{n-1} = 2 \operatorname{Im}|\overline{S(\omega)}\, \omega S'(\omega)| \neq 0$. *We then can choose λ such that*

$$A_{n-1} + n\lambda A_n \neq 0. \tag{10}$$

From (8) and Lemma 2 we have

LEMMA 5. *If $n = 1$, $S(Z) \in \mathscr{S}(1)$, then*

$$\begin{aligned} Z(S, \Gamma) &\leqslant r, && r \text{ odd},\\ &\leqslant r-1, && r \text{ even}. \end{aligned}$$

From the proof of Theorem 2 and (8), we have

LEMMA 6. *If $S(Z) \in G_n(\Gamma)$, then $Z(S, \Gamma) \leqslant r+1$.*

COROLLARY. *If $S(Z) \in G_n(\Gamma)$, then the total number of zeros of the function $T_\omega(x)$ defined by (7) can be estimated as $Z(T_\omega, E_1) \leqslant r+1$.*

LEMMA 7. *Let $T_\omega(x)$ be defined by (7), ω satisfy (9), $n \geqslant 2$. If $T_\omega(x)$ has no zero with multiplicity less than 2, then*

$$\begin{aligned} Z(T_\omega, E_1) &\leqslant r, && \text{if} \quad r, n \text{ odd or } r, n \text{ even},\\ &\leqslant r-1, && \text{otherwise}. \end{aligned}$$

Proof. In view of the hypothesis, it is easy to prove that there are two knots x_k, x_l such that $T(x_k) \neq 0$, $T(x_l) \neq 0$. Since $A_k \neq 0$, $A_{k-1} \neq 0$, we may choose $\lambda = x_k$ (or x_l) satisfying (10); since λ is a knot, we obtain from Theorem 2 that $Z(T_\omega, E_1) = Z(R, E_1) \leqslant r$. If n is odd (even) $T_\omega(x)$ has an odd (even) number of sign changes, so Lemma 7 is proved. Q.E.D.

LEMMA 8. *If $S(Z) \in G_n(\Gamma)$, $S(Z)$ has a simple zero ξ, ξ is a knot, then $Z(S, \Gamma) \leqslant r-1$.*

Proof. With no loss of generality, we assume $\xi = Z_1$. Under a linear transformation $l_{Z_1}(x)$, we get the real spline $T(x) = \mathrm{Re}\{(-2iZ_1)^{-1}\overline{g(Z_1)}(x+i)^n S(Z(x))\}$, where $g(x)$ is a polynomial of degree $n-1$. $T(x)$ has knots $\{x_j\}_2^r$, and $T(x)$ is a polynomial of degree $n-1$ on $(-\infty, x_2)$ and (x_r, ∞).

Case 1.

$$T^{(n)}(x_r-) \neq 0, \qquad T^{(n)}(x_2+) \neq 0.$$

Let λ be a point on the real x axis satisfying

$$\text{(a)} \quad T(\lambda) \neq 0, \qquad \text{(b)} \quad \lambda < x_2 - L,$$

where $L = \mathrm{Max}\{1, 2^n(2n-1)!\, M/m\}$, $M = \mathrm{Max}_{1 \leqslant j \leqslant n-1} |T^{(j)}(x_2)|$, $m = |T^{(n-1)}(x_2)|$. Let $R(x) = T(x)/(x-\lambda)^n$, since $T^{(n-1)}(x_2) > 0$, $T^{(n)}(x_2-) = T^{(n)}(x_r+) = 0$. Therefore, $S^+((-1)^l R^{(l)}(x_2+))_{n-1}^n + S^+((R^{(l)}(x_2-))_{n-1}^n = 1 + S^+(-T^{(n-1)}(x_2), T^{(n)}(x_2+))$.

Following the proof of Theorem 2, we have

$$Z(R, I_0 \cup I_1) = r - 1 - \sum h_j - \sum H_j, \qquad h_2 \geqslant 1, \quad h_j \geqslant 0, \quad H_j \geqslant 0.$$

Hence $Z(R, I_0 \cup I_1) \leqslant r-2$, $Z(T, E_1) \leqslant r-2$; since $S(Z)$ has a zero at Z_1, then $S(Z, \Gamma) \leqslant r-1$.

Case 2.

$$T^{(n)}(x_2+) = 0, \qquad T^{(n)}(x_3+) = 0.$$

We choose λ satisfying

$$\text{(c)} \quad T(\lambda) \neq 0, \qquad \text{(d)} \quad \lambda < x - K,$$

where $K = \mathrm{Max}\{1, 2^n(2n-1)!\, M_1/m, \quad 2^n(2n-1)!\, M_1/m_2, \quad 2^n(2n-1)!\, M_2/m_1\}$, $M_1 = \mathrm{Max}_{1 \leqslant j \leqslant n-1} |T^{(j)}(x_3)|$, $M_2 = \mathrm{Max}_{1 \leqslant j \leqslant n-1} |T^{(j)}(x_2)|$, $m_1 = |T^{(n-1)}(x_2)|$, $m_2 = |T^{(n)}(x_3+)|$. We then have $h_2 = S^+((-1)^l R^{(l)}(x_2+))_{n-1}^n + S^+(R^{(l)}(x_2-))_{n-1}^n = 1$, $h_3 = S^+((-1)^l R^{(l)}(x_2+))_{n-1}^n + S^+(R^{(l)}(x_2))_{n-1}^n \geqslant 1$; therefore, $Z(R, I_0 \cup I_1) \leqslant r-3$, $Z(T, E_1) \leqslant r-3$, $Z(S, \Gamma) \leqslant r-2$. The remaining cases can be treated in the same way.

LEMMA 9. *Let $S(Z) \in G_n(\Gamma)$. If $S(Z)$ has a simple zero ξ, ξ is not a knot, then*

$$Z(S, \Gamma) \leqslant r - 1, \qquad \textit{if } n \textit{ is even or if } n \textit{ odd and } r \textit{ even},$$
$$\leqslant r, \qquad n, r \textit{ odd}.$$

Proof. With no loss of generality, let ξ belong to the interior of Γ_r. Under a linear transformation $l_\xi(x): Z = \xi(x-i)/(x+i)$, we obtain a real spline $T(x)$ with knots $\{x_j\}_1^r$, and $T(x) = \sum_0^{n-1} A_k x^k$, $x \in \gamma_0 \cup \gamma_r$, $A_{n-1} > 0$. We choose λ, $T(\lambda) \neq 0$, using the method in the proof of Lemma 8 (but here we take x_1 instead of x_2) we have $h_1 \geqslant 1$ and

$$Z(R, I_0 \cup I_1) \leqslant r - \sum_1^r h_j - \sum H_j \leqslant r - 1,$$

$$Z(T, E_1) \leqslant r - 1.$$

By using the definition of the multiplicity of zero of real splines it is easy to verify the assertion of Lemma 9 for the following three cases: (i) n odd, r odd, (ii) n odd, r even, (iii) n even, r odd.

Now suppose n, r are even. We then choose λ such that the main term in the expansion (5) for $R^{(l)}(x)$ is one which contains the highest derivative of $T(x)$ with respect to x when $x \in [x_1, x_k]$.

Since $T^{(n-1)}(x)$ is a linear function of x on γ_j ($\gamma_j = [x_j, x_{j+1})$), $0 \leqslant j \leqslant r$, $x_0 = -\infty$, $x_{r+1} = +\infty$, $T^{(n-1)}(x_1)\, T^{(n-1)}(x_r) = ((n-1)!\, A_{n-1})^2 > 0$. $T^{(n-1)}(x)$ has an even number of sign changes, since r is even, thus there is one interval r_k where $T^{(n-1)}(x)$ does not change sign. We may assume $T^{(n-1)}(x) > 0$, $x \in r_k$; hence

$$T^{(n-1)}(x_{k-1}) > 0, \qquad T^{(n-1)}(x_k) > 0. \tag{11}$$

By the choice of λ, we assert $R^{(n)}(x_{k-1}-) \neq 0$ and $R^{(n)}(x_k+) \neq 0$. From the proof of Theorem 2 we now have the following expression

$$\begin{aligned} &W_\varepsilon(x_{k-1}) + W_\varepsilon(x_k) \\ &\quad = (\alpha_{k-1} - 1) + S^+(R^{(n-1)}(x_{k-1}), R^{(n)}(x_{k-1}-)) + (\alpha_k - 1) \\ &\qquad + S^+(R^{(n-1)}(x_k), R^{(n)}(x_k-)) + S^+(-R^{(n-1)}(x_{k-1}), R^{(n)}(x_{k-1}+)) \\ &\qquad + S^+(-R^{(n-1)}(x_k), R^{(n)}(x_k+)). \end{aligned} \tag{12}$$

If $T^{(n)}(x_{k-1}+) \neq 0$, the choice of λ yields $\operatorname{sign}(R^{(n)}(x_{k-1}+)) = \operatorname{sign}(T^{(n)}(x_{k-1}+)) = \operatorname{sign}(T^{(n)}(x_k-)) = \operatorname{sign}(R^{(k)}(x_k-)) \neq 0$, from (11), $R^{(n-1)}(x_{k-1})\, R^{(n-1)}(x_k) > 0$; hence

$$W_\varepsilon(x_{k-1}) + W_\varepsilon(x_k) \geqslant \alpha_{k-1} + \alpha_k - 1. \tag{13}$$

But $\operatorname{sign}(R^{(n)}(x_1+)) = \operatorname{sign}(T^{(n)}(x_1+))$, $\operatorname{sign}(R^{(n)}(x_1-)) = \operatorname{sign}(-T^{(n-1)}(x_1))$, $\operatorname{sign}(R^{(n-1)}(x_1)) = \operatorname{sign}(T^{(n-1)}(x_1))$, so that

$$\begin{aligned} S^{+}((-1)^{l} R^{(l)}(x_1+))_{n-1}^{n} + S^{+}(R^{(l)}(x_1-))_{n-1}^{n} &= S^{+}(-T^{(n-1)}(x_1), T^{(n)}(x_1+)) + S^{+}(T^{(n-1)}(x_1), -T^{(n-1)}(x_1)) \\ &= S^{+}(-T^{(n-1)}(x_1), T^{(n)}(x_1+)) + 1. \end{aligned} \tag{14}$$

From the proof of Theorem 2, we have

$$W_{\varepsilon}(x_1) \geqslant \alpha_1. \tag{15}$$

From (13), (15), if $k > 2$, we conclude from (13), (15)

$$W_{\varepsilon}(x_1) + W_{\varepsilon}(x_{k-1}) + W_{\varepsilon}(x_k) + 1 \geqslant \alpha_1 + \alpha_{k-1} + \alpha_k,$$

namely, $h_1 + h_{k-1} + h_k \geqslant 2$. We then have $Z(R, I_0 \cup I_1) \leqslant r - 2$.

If $k = 2$, then from (12), (14) we have $\alpha_1 + \alpha_2 \leqslant W_{\varepsilon}(x_1) + W_{\varepsilon}(x_2)$, namely, $h_1 + h_2 \geqslant 2$. Then from the proof of Theorem 2, we infer that $Z(R, I_0 \cup I_1) \leqslant r - 2$.

The remaining cases can be treated in the same way, so that we still have $Z(R, I_0 \cup I_1) \leqslant r - 2$; then $Z(T, E_1) \leqslant r - 1$, hence $Z(S, \Gamma) \leqslant r - 1$. Q.E.D.

From Lemmas 3, 5, 7, and 8 and (8), Theorem 3 follows directly.

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